METU - NCC

Differential Equations Midterm 1	
Code : *MAT 219* Acad.Year : *2015-2016* Semester : *Fall* Date : *15.11.2015* Time : *9:40* Duration : *110 min*	Last Name: List #: Name : KEY Student # : Signature :
	6 QUESTIONS ON 5 PAGES TOTAL 100 POINTS

1. (24)	2. (6)	3. (16)	4. (14)	5. (20)	6. (20)	

Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

1. *(12×2=24pts)* Solve the following differential equations:

(A) Solve as a separable equation: $y' + 2ty = 0$, where $y(1) = -\frac{3}{e}$.

$\frac{dy}{dt} = -2ty \Rightarrow \frac{dy}{y} = -2t\,dt,\ y \neq 0 \Rightarrow \ln|y| = -t^2 + C \Rightarrow$

$\Rightarrow |y| = Ce^{-t^2},\ C>0 \Rightarrow y = Ce^{-t^2},\ C \neq 0.$

But $y = 0$ is a solution, therefore

$\boxed{y = Ce^{-t^2}}$ But $y(1) = -\frac{3}{e} \Rightarrow -3e^{-1} = Ce^{-1}$

$\Rightarrow C = -3 \Rightarrow \boxed{y = -3e^{-t^2}}$

(B) Solve: $\frac{1}{2t}y' + y = e^{-t^2}$. First convert $y' + 2ty = 2te^{-t^2}$, $t \neq 0$.

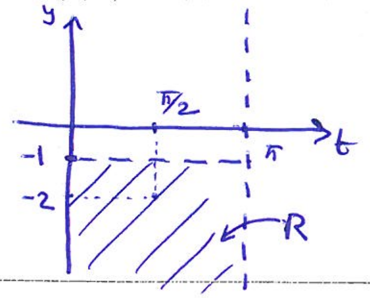
An integrating factor $\mu(t) = e^{\int 2t\,dt} = e^{t^2}$. Then

$(e^{t^2}y)' = 2t \Rightarrow e^{t^2}y = t^2 + C \Rightarrow y = Ce^{-t^2} + t^2e^{-t^2}$

2. *(6pts)* Sketch a rectangular region in the ty-plane where the hypotheses of the Existence and Uniqueness Theorem are applicable to the following IVP: $y' = \cot(t)y^2(1+y)^{-1}$, $y(\pi/2) = -2$.

Put $R = (0, \pi) \times (-\infty, -1)$ and $f(t,y) = \dfrac{y^2 \cot(t)}{1+y}$.

Note that $\dfrac{\partial f}{\partial y} = \dfrac{2y\cot(t)(1+y) - y^2\cot(t)}{(1+y)^2} = \dfrac{y\cot(t)(2+y)}{(1+y)^2}$. Thus $f, \dfrac{\partial f}{\partial y} \in C(R)$ and Exist-Uniq-Th is applicable to the region R.



3. *(12+4=16pts)* The following two problems are about exact differential equations.

(A) Show that (i) the following differential equation is exact, then (ii) solve it:

$$(2t^2y + 2y)y' + (2ty^2 + 2t) = 0$$

$(2ty^2 + 2t)dt + (2t^2y + 2y)dy = 0$, $M_y = 4ty = N_t$.

There exists a potential function $\Psi(t,y)$ such that

$$\begin{cases} \Psi_t = 2ty^2 + 2t \longrightarrow \Psi = t^2y^2 + t^2 + C(y) \\ \Psi_y = 2t^2y + 2y = 2t^2y + C'(y) \longrightarrow C'(y) = 2y \longrightarrow C(y) = y^2 \end{cases}$$

$$\boxed{t^2y^2 + t^2 + y^2 = c}$$

(B) Find an integrating factor either $\mu = \mu(t)$ or $\mu = \mu(y)$ to make the following differential equation exact:

$$dt + (t/y - \cos(y))dy = 0.$$

DO NOT SOLVE THE DIFFERENTIAL EQUATION (only find μ).

Put $\mu(t,y) = \mu(y)$. Then $\mu(y)dt + \mu(y)(t/y - \cos(y))dy = 0$

and $\mu'(y) = \mu(y)\dfrac{1}{y}$ or $\dfrac{d\mu}{\mu} = \dfrac{dy}{y} \Rightarrow \boxed{\mu(y) = y}$.

Now

$$y\,dt + (t - y\cos(y))dy = 0$$

is an exact equation.

(Extra space for # 3 (B)...)

4. *(8+6=14pts)* A tank originally contains 100 L of pure water. Then water containing 1/2 kg of salt per liter is poured into the tank at a rate of 2 L/min, and the mixture is allowed to leave at the same rate.

Write a differential equation **with initial values** for $Q(t)$ = the amount of salt in the tank at time t.

DO NOT SOLVE THE DIFFERENTIAL EQUATION.

$$\frac{dQ}{dt} = \frac{1}{2} \times 2 - 2 \times \frac{Q(t)}{100} = 1 - \frac{Q(t)}{50}$$

$$Q(0) = 0.$$

The solution to the differential equation above is $Q(t) = 50(1 - e^{-t/50})$.

After 10 min the process is stopped, and pure water is poured into the tank at a rate of 2 L/min, with the mixture again leaving at the same rate.

Write a new differential equation **with initial values** for $R(t)$ = the amount of salt in the tank at time t for $t \geqslant 10$.

DO NOT SOLVE THE DIFFERENTIAL EQUATION

$$\frac{dR}{dt} = 0 \times 2 - 2 \times \frac{R(t)}{100} = -\frac{1}{50} R(t)$$

Hence we have IVP

$$\begin{cases} R' = -\frac{1}{50} R \\ R(0) = Q(10) = 50(1 - e^{-1/5}) \end{cases}$$

5. *(10+10=20pts)* Calculate the eigenvalues and eigenvectors of the following matrices.

(A) $\begin{bmatrix} 7 & -9 \\ 3 & -5 \end{bmatrix}$ $\Delta(\lambda) = \begin{vmatrix} 7-\lambda & -9 \\ 3 & -5-\lambda \end{vmatrix} = (\lambda-7)(\lambda+5)+27 = \lambda^2 - 2\lambda - 8 =$

$= (\lambda+2)(\lambda-4)$, $\sigma(A) = \{-2, 4\}$.

$A+2 = \begin{bmatrix} 9 & -9 \\ 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, $V_{-2} = \ker(A+2) = \{x=y\} = \operatorname{Span}\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$

$A-4 = \begin{bmatrix} 3 & -9 \\ 3 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$, $V_4 = \ker(A-4) = \{x = 3y\} = \operatorname{Span}\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\}$

So, $\vec{f}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{f}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

(B) $\begin{bmatrix} 7 & 0 & -9 \\ 0 & 2 & 0 \\ 3 & 0 & -5 \end{bmatrix}$, $\Delta(\lambda) = \begin{vmatrix} 7-\lambda & 0 & -9 \\ 0 & 2-\lambda & 0 \\ 3 & 0 & -5-\lambda \end{vmatrix} = -(\lambda-7)(\lambda-2)(\lambda+5) + 27(2-\lambda)$

$= -(\lambda-2)\left((\lambda-7)(\lambda+5)+27\right)$

$= -(\lambda-2)(\lambda+2)(\lambda-4)$

$\sigma(A) = \{-2, 2, 4\}$

$\lambda_1 = -2 \Rightarrow A+2 = \begin{bmatrix} 9 & 0 & -9 \\ 0 & 4 & 0 \\ 3 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\vec{f}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_2 = 2 \Rightarrow A-2 = \begin{bmatrix} 5 & 0 & -9 \\ 0 & 0 & 0 \\ 3 & 0 & -7 \end{bmatrix}$, $\vec{f}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_3 = 4 \Rightarrow A-4 = \begin{bmatrix} 3 & 0 & -9 \\ 0 & -2 & 0 \\ 3 & 0 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\vec{f}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

6. *(8+6+6=20pts)* This problem has three parts about solutions to three **different** systems.

(A) Write the general solution to the 2 × 2 system of linear differential equations $\vec{x}\,' = \mathrm{A}\,\vec{x}$ if the matrix A has

- eigenvector $\vec{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}$ with eigenvalue $\lambda_1 = 3$,
- eigenvector $\vec{v}_2 = \begin{bmatrix}1\\0\end{bmatrix}$ with eigenvalue $\lambda_2 = -1$.

$$\Psi(t) = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & 0 \end{bmatrix}$$

$\vec{x}(t) = \Psi(t)\,\vec{c}$ is the general solution.

(B) If the system of differential equations $\vec{x}\,' = \mathrm{B}\,\vec{x}$ has general solution

$$x_1(t) = 3c_1e^{4t} + 2c_2$$
$$x_2(t) = c_2$$

then what are the eigenvalues and eigenvectors of the matrix B?

$$\vec{x}(t) = \begin{bmatrix} 3e^{4t} & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Rightarrow \Psi(t) = \begin{bmatrix} 3e^{4t} & 2 \\ 0 & 1 \end{bmatrix}$$

Therefore $\lambda_1 = 4$, $\lambda_2 = 0$ and $\vec{f}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\vec{f}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(C) What is the solution to the initial value problem $\vec{x}\,' = \mathrm{C}\,\vec{x}$ with $\vec{x}(10) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ if the general solution to the differential equation is

$$\vec{x}(t) = c_1\begin{bmatrix} -1 \\ 2 \end{bmatrix}e^{t} + c_2\begin{bmatrix} 3 \\ -5 \end{bmatrix}e^{-3t} = \begin{bmatrix} -e^{t} & 3e^{-3t} \\ 2e^{t} & -5e^{-3t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

For $t=10$ we have

$$\begin{bmatrix} -e^{10} & 3e^{-30} \\ 2e^{10} & -5e^{-30} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} -e^{10} & 3e^{-30} & 2 \\ 2e^{10} & -5e^{-30} & -1 \end{array}\right] \overset{R_2+2R_1}{\sim}$$

$$\sim \left[\begin{array}{cc|c} -e^{10} & 3e^{-30} & 2 \\ 0 & e^{-30} & 3 \end{array}\right] \overset{R_1-3R_2}{\sim} \left[\begin{array}{cc|c} -e^{10} & 0 & -7 \\ 0 & e^{-30} & 3 \end{array}\right], \quad \begin{aligned} c_1 &= 7e^{-10} \\ c_2 &= 3e^{30} \end{aligned}$$

$$\vec{x}(t) = \begin{bmatrix} -e^{t} & 3e^{-3t} \\ 2e^{t} & -5e^{-3t} \end{bmatrix}\begin{bmatrix} 7e^{-10} \\ 3e^{30} \end{bmatrix} = \begin{bmatrix} -7e^{t-10} + 9e^{30-3t} \\ 14e^{t-10} - 15e^{30-3t} \end{bmatrix}.$$